

# CRITICAL SLOWING DOWN AND DEFECT FORMATION

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The formation of topological defects in a second order phase transition in the early universe is an out-of-equilibrium process. Condensed matter experiments seem to support Zurek's mechanism, in which the freezing of thermal fluctuations close to the critical point (critical slowing down) plays a crucial role. We discuss how this picture can be extrapolated to the early universe, pointing out that new scaling laws may emerge at very high temperatures and showing how critical slowing down emerges in the context of a relativistic quantum field theory.

## 1 Zurek's picture of defect formation

The formation of topological defects (domain walls, strings,...) in a second order phase transition is a common phenomenon in condensed matter and cosmology. One relevant point in this context is to determine the initial correlation length of the pattern of defects emerging from the critical region or, in other words, to answer the question: 'when symmetry breaks, how big are the smallest identifiable pieces?'<sup>1</sup>.

Historically, the first answer to this question<sup>2</sup> was based on the thermal activation mechanism. In this picture, the pattern of defects stabilizes at the Ginzburg temperature,  $T_G$ , below which thermal fluctuations are unable to overcome the free energy barrier between inequivalent vacua. The initial length scale of the pattern of topological defects is then given by the correlation length at the Ginzburg temperature,  $\xi(T_G)$ , which can be estimated as

$$\xi(T_G) \sim \frac{1}{\lambda^{1/2}\mu} \quad (1)$$

where  $\mu$  is the mass scale and  $\lambda$  the coupling constant of the theory. The relevant point about eq. (1) is its independence on the rate at which temperature is changed (quench rate), due to the use of the *equilibrium* free energy from the critical temperature  $T_C$  down to  $T_G$ .

More recently Zurek has proposed a new picture of defect formation<sup>1</sup>, in which dynamical aspects of the phase transition play a key role. The main ingredient is *critical slowing down* (CSD), *i.e.* the vanishing of the damping rate of thermal fluctuations close to the critical point. CSD can be discussed in the context of a non-relativistic, classical scalar theory described by an order

parameter  $\eta(t, \vec{r})$  and a Langevin-type equation of motion as

$$\frac{\partial \eta(t, \vec{r})}{\partial t} = -\Gamma \frac{\delta \mathcal{F}}{\delta \eta(t, \vec{r})} + \zeta(t, \vec{r}), \quad (2)$$

where  $\mathcal{F}$  is the free-energy,  $\Gamma$  a phenomenological parameter, and  $\zeta(t, \vec{r})$  a white noise term. Eq. (2) has damped plane wave solutions of the form  $\eta(t, \vec{r}) \sim e^{i \vec{k} \cdot \vec{r}} e^{-\gamma_k t}$ , with  $\gamma_k = \Gamma[|\vec{k}|^2 + 2\alpha(T - T_C)]$ . CSD is the statement that long-wavelength fluctuations ( $\vec{k} \rightarrow 0$ ) are not damped as  $T \rightarrow T_C$ , which is clearly realized here.

What are the consequences of CSD on the formation of topological defects? The scaling  $\tau_0 = 1/\gamma(T) \sim (T - T_C)^{-1}$  (where we have defined  $\gamma \equiv \gamma_{\vec{k}=0}$ ) - obtained from the simple model (2) - can be generalized to

$$\xi \sim \varepsilon^{-\nu}, \quad \tau_0 \sim \varepsilon^{-\mu} \quad \text{for } \varepsilon \rightarrow 0, \quad (3)$$

where  $\varepsilon = |(T - T_C)/T_C| = |(t - t_C)/\tau_Q|$  measures the distance in temperature - or in time - from the critical point. As  $t_C$  is approached the relaxation time grows until a certain time  $t^*$  at which it becomes larger than the time left before the phase transition, *i.e.*  $\tau_0(t^*) = (t_C - t^*)$ . Thermal fluctuations generated from  $t^*$  to  $t_C$  are then unable to relax before  $t_C$  and the system cannot follow the equilibrium effective potential. Then, as long as the quench-time  $\tau_Q$  is finite -as in any practical application- a second order phase transition takes place out of thermal equilibrium.

To modify the thermal activation picture, Zurek's proposed that the relevant length scale of the pattern of topological defects was given by the correlation length at  $t^*$ . Using the scaling laws in (3) this implies a quench-time dependence of  $\xi(t^*)$  as

$$\xi(t^*) \sim \tau_Q^{\nu/(1+\mu)}, \quad (4)$$

to be compared with eq. (1).

A recent generation of solid-state experiments, on phase transitions in liquid crystals and in  $^3\text{He}$  and  $^4\text{He}$ , have ruled out the  $\tau_Q$  independent scaling law (1) and are compatible with Zurek's eq. (4). The question is then how much of these results can be extrapolated to the relativistic high temperature environment of the early universe.

## 2 Critical slowing down in the early universe ?

In order to extend Zurek's picture to the early universe, two main facts have to be taken into account; the expansion of the universe, and the need to use a

relativistic quantum field theory (QFT), since temperatures are typically much larger than the masses of the particles.

In a radiation dominated universe the quench time is given by  $\tau_Q = -(\dot{T}/T)^{-1} = 2\tau_H$ , where  $\tau_H$  the inverse of the Hubble parameter, which can be taken as a measure of the age of the universe.

$\tau_H$  is the third relevant time-scale of the problem, besides the time to the phase transition,  $t_C - t = \tau_Q \varepsilon$ , and the relaxation time  $\tau_0$ . If, at  $\varepsilon = 1$ ,  $\tau_0 < 2\tau_H$ , the picture will be similar to that discussed in the previous paragraph, with  $\tau_0$  growing larger than  $(t_C - t)$  at some time  $t^*$ , leading to the scaling law of eq. (4). There is however a second possibility, namely that the lifetime of fluctuations becomes larger than the age of the universe, which happens if  $\tau_0(\varepsilon = 1) > 2\tau_H$ . If this happens, the time  $t^*$  is not relevant any more since what counts is the time  $t_{\text{age}}$  when  $\tau_0 = 2\tau_H$ . A different scaling law is obtained in this case, namely

$$\xi(t_{\text{age}}) \sim \tau_Q^{\nu/\mu}. \quad (5)$$

Which of the two scaling laws is realized depends on the epoch at which the phase transition takes place. Typically, taking a  $\lambda\Phi^4$  theory with  $\lambda = 10^{-2}$ , eq. (4) ( eq. (5)) is realized for  $T < 10^{11}\text{GeV}$  ( $T > 10^{11}\text{GeV}$ ).

The need to use relativistic QFT poses more subtle questions. First of all, we have to identify the relaxation rate. Instead of relying on phenomenological equations of motion like (2), first principles equations- directly derived from the QFT- have to be employed. They have the general form

$$\left[ \frac{\partial^2}{\partial t^2} + |\vec{k}|^2 + m^2 + \text{Re}\Pi(E_k, \vec{k}) + 2\gamma_{\vec{k}} \frac{\partial}{\partial t} + \dots \right] \Phi(t, \vec{k}) = \zeta(t, \vec{k}) + \dots, \quad (6)$$

where  $\Pi(E_k, \vec{k})$  is the self-energy,

$$\gamma_{\vec{k}} = \frac{\text{Im}\Pi(E_k, \vec{k})}{2E_k}, \quad E_k^2 = |\vec{k}|^2 + m^2 + \text{Re}\Pi(E_k, \vec{k}), \quad (7)$$

and the ellipses represent terms which are non-local in time (memory terms). Now the damped plane wave solutions have the form  $\Phi \sim e^{-i(E_k t - \vec{k} \cdot \vec{x})} e^{-\gamma_k t}$ . Comparing with (2) we see that the equation of motion in this case is second order in time and, moreover, the dissipative term is proportional to the *imaginary part* of the self-energy, and not to the real part, as for eq. (2). At first sight, it is then not obvious at all whether CSD is realized in the relativistic QFT as well.

Indeed,  $\gamma$  has been computed in perturbation theory by Parwani<sup>3</sup>. At

two-loops in the hard-thermal-loop resummed theory one gets

$$\gamma_{p.t.} = \frac{1}{1536\pi} l_{qu} \lambda^2 T^2 \quad (8)$$

where  $l_{qu} = 1/m(T)$  is the Compton wavelength. The above result is valid as long as  $T$  is much larger than any mass scale of the  $T = 0$  theory. If at  $T = 0$  there is spontaneous symmetry breaking  $m(T)$  has the form  $m^2(T) = -\mu^2 + \lambda T^2/24$ , thus giving a critical temperature  $T_C^2 = 24\mu^2/\lambda$ . As  $T \rightarrow T_C$ ,  $m(T) \rightarrow 0$  and eq. (8) diverges. In other words, the resummed perturbation theory result is completely at odds with what expected; we have critical speeding up instead of slowing down!

### 3 RG computation of $\gamma$

It is well known that (resummed) perturbation theory cannot be trusted close to the critical point, due to the divergence of its effective expansion parameter, *i.e.*  $\lambda T/m(T)$ . In refs.<sup>4,5</sup> it was shown that the key effect which is missed by perturbation theory is the dramatic thermal renormalization of the coupling constant, which vanishes in the critical region. The running of the coupling constant for  $T \simeq T_C$  is crucial also in turning the divergent behavior of eq. (8) into a vanishing one.

The details of the computation of  $\gamma(T)$  in the framework of the Thermal Renormalization Group (TRG) of ref.<sup>5</sup> can be found in<sup>6</sup>.

In Fig. 1 we plot the results for the temperature-dependent damping rate  $\gamma$  and coupling constant, as a function of the temperature<sup>6</sup>. The dashed line has been obtained by keeping the coupling constant fixed to its  $T = 0$  value ( $\lambda = 10^{-2}$ ), and reproduces the divergent behavior found in perturbation theory (eq. (8)). The crucial effect of the running of the coupling constant is seen in the behavior of the dot-dashed line. For temperatures close enough to  $T_C$ , the coupling constant (solid line in Fig.2) is dramatically renormalized and it decreases as

$$\lambda(T) \sim \varepsilon^\nu$$

where  $\nu \simeq 0.53$  in our approximations. The mass also vanishes with the same critical index. The decreasing of  $\lambda$  drives  $\gamma$  to zero, but with a different scaling law<sup>6</sup>,

$$\gamma(T) \sim \varepsilon^\nu \log \varepsilon.$$

Taking couplings bigger than the one used in this letter ( $\lambda = 10^{-2}$ ), the deviation from the perturbative regime starts to be effective farther from  $T_C$ . Defining an effective temperature as  $\lambda(T)/\lambda \leq 1/2$  for  $T_C < T \leq T_{eff}$  we find that  $t_{eff}$  scales roughly as  $t_{eff} \sim \lambda$ .

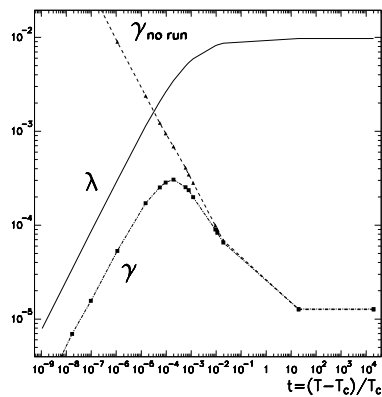


Figure 1: Temperature dependence of the coupling constant (solid line), and of the damping rate with the effect of the running of  $\lambda$  included (dash-dotted) and excluded (dashed). The values for  $\gamma$  have been multiplied by a factor of 10.

#### 4 Conclusion

We have seen that Zurek's picture of second order phase transitions is basically valid also in the early universe, provided non-perturbative methods are employed in order to reproduce CSD. This problem seems to be tailored on the *real-time* TRG method of ref. <sup>5</sup> better than on any other computation method. Indeed, as we have seen, perturbation theory fails close to  $T_C$ . Moreover, since we have to compute a non-static quantity (*i.e.* at non-zero external energy, see eq. (7)), neither lattice simulations nor the Exact RG of the second of refs. <sup>4</sup> -which are implemented in euclidean time- can be employed here.

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